

Symplectic embedding of a massive vector–tensor theory with topological coupling

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Abstract. In the symplectic Lagrangian framework we in a new fashion embed an irreducible massive vector–tensor theory into a gauge invariant system, which has become reducible, by extending the configuration space to include an additional pair of scalar and vector fields, which give the desired Wess–Zumino action. A comparison with the BFT Hamiltonian embedding approach is also given.

1 Introduction

The Dirac quantization method [1] has been widely used in order to quantize a Hamiltonian system involving first- and second-class constraints. However, the resulting Dirac brackets may be field-dependent and non-local, and thus pose serious ordering problems. The BRST quantization [2] along the lines of Dirac established by Batalin, Fradkin, and Vilkovisky [3], and then improved in a more tractable and elegant Hamiltonian embedding by Batalin, Fradkin, and Tyutin (BFT) [4], does not suffer from these difficulties, as it relies on a simple Poisson bracket structure. As a result, this embedding of a second-class system into first-class one has received much attention in the past decade [5–7].

While the various quantization methods based on the Hamiltonian formulation have been developed for general types of gauge theories, Faddeev and Jackiw [8] have introduced an equivalent scheme based on a first order Lagrangian which does not need to classify constraints as primary, secondary, etc. Since this scheme deals with a smaller number of constraints than that of Dirac, it has proved to be relatively simple for using it to find the Dirac brackets. After this work, there were numerous related analyses [9–11]. In particular, we had shown that, in this framework, a gauge non-invariant theory can be embedded into a gauge invariant one by investigating the properties of the symplectic two-form matrix and its corresponding zero modes [12]. Recently, there has occurred some renewed interest [13–15] in the subject of symplectic embedding including non-commutative theories.

On the other hand, antisymmetric tensor fields, which first appeared as a mediator of the string interaction [16], have lead to much interest in an alternative of the Higgs mechanism [17,18]. With the topologically interacting term of the form $B \wedge F$, the physical degree of freedom of the antisymmetric rank two tensor field B is absorbed by the vector field, making it massive. This mechanism is considered generic in string phenomenology [19]. Moreover, various dual descriptions between different theories have been widely studied where the antisymmetric tensor field plays an important role in the realization of dualities [20,21]. These dual relations were also independently confirmed by defining dual operations on the space of pairs of different gauge forms [22]. There was the surprising result that, like the well-known dual equivalence [23] of the first order self-dual theory and Maxwell–Chern–Simons theory in $d = 3$, they have all shown that the first order master action even in $d = 4$ has a dual relation with the Maxwell–Kalb–Ramond (MKR) theory. Recently, it has been shown that through the BFT embedding technique the gauge non-invariant master action is equivalent to the gauge invariant MKR theory [24]. Very recently, we have generalized their results to include both the gauge symmetry breaking term and the topological coupling one [7], resulting in a new type of Wess–Zumino (WZ) action [25] as well as the usual Stückelberg one.

This paper is organized as follows. In Sect. 2, we briefly show the dual relation of the first order master Lagrangian between the Abelian Proca and the Kalb–Ramond (KR) massive theories classically by using the equations of motion. Then, we explicitly carry out the symplectic quantization for this gauge non-invariant Lagrangian. In Sect. 3, we embed this master Lagrangian into the desired gauge invariant one, and we make clear the relation between the “trivial” zero modes and the symmetry transformations.

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We also explicitly show that one of the “trivial” zero modes is related to a reducible constraint. In Sect. 4, we compare the result of the symplectic embedding with the BFT one. Our conclusion is given in Sect. 5.

2 Symplectic quantization of the master Lagrangian

In this section, we reconsider the first order Lagrangian [21] for the Abelian massive vector and tensor fields, and quantize it explicitly through the symplectic scheme. This Lagrangian is not only known to have a dual relation with the MKR Lagrangian [7, 21, 22, 24], but also plays the role of a master Lagrangian of the Proca and the KR models.

First, let us briefly review that the following first order Lagrangian

$$\mathcal{L} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{1}{2}A_\mu A^\mu + \frac{1}{2m}\epsilon_{\mu\nu\rho\sigma}B^{\mu\nu}\partial^\rho A^\sigma \quad (2.1)$$

describes classically the massive Proca and KR theories simultaneously, i.e., the Lagrangian (2.1) is a master Lagrangian¹ of the mentioned two theories. By varying the Lagrangian with the fields A^μ and $B^{\mu\nu}$, one obtains their equations of motion as follows:

$$\begin{aligned} B_{\mu\nu} - \frac{1}{m}\epsilon_{\mu\nu\rho\sigma}\partial^\rho A^\sigma &= 0, \\ A_\mu + \frac{1}{2m}\epsilon_{\mu\nu\rho\sigma}\partial^\nu B^{\rho\sigma} &= 0, \end{aligned} \quad (2.2)$$

respectively. By eliminating the antisymmetric $B^{\mu\nu}$ fields from the Lagrangian (2.1), the Proca model is obtained as follows:

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{4m^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}A_\mu A^\mu. \quad (2.3)$$

Similarly, by using the equation of motion for the field A^μ , one could obtain the KR Lagrangian as follows:

$$\mathcal{L}_{\text{KR}} = \frac{1}{12m^2}G_{\mu\nu\sigma}G^{\mu\nu\sigma} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}. \quad (2.4)$$

This shows that the first order Lagrangian (2.1) is the master Lagrangian of the two massive theories at the classical level.

On the other hand, according to the symplectic scheme [8, 9], it is easy to find their symplectic brackets in a few steps equivalent to the Dirac ones in the Hamiltonian formulation:

$$\begin{aligned} \{A^0(x), A^i(y)\} &= \partial_x^i \delta(x-y), \\ \{A^i(x), \pi_j(y)\} &= m^2 \delta_j^i \delta(x-y) \end{aligned} \quad (2.5)$$

for the Proca model [14], as well as

$$\{B^{0i}(x), B^{jk}(y)\} = (\delta^{ij}\partial_x^k - \delta^{ik}\partial_x^j)\delta(x-y),$$

¹ For a quantum mechanical treatment, see [21].

$$\{B^{ij}(x), \pi_{kl}(y)\} = m^2(\delta_k^i\delta_l^j - \delta_k^j\delta_l^i)\delta(x-y) \quad (2.6)$$

for the KR theory [26].

Now, in order to implement the usual symplectic quantization of the master Lagrangian itself which is composed of two kinds of fields with different ranks as well as their topological coupling term, we rewrite the Lagrangian as an alternating first ordered one from the symmetrized form of (2.1):

$$\mathcal{L}^{(0)} = \frac{1}{4m}\epsilon_{ijk}B^{jk}\dot{A}^i - \frac{1}{4m}\epsilon_{ijk}A^k\dot{B}^{ij} - \mathcal{H}^{(0)}, \quad (2.7)$$

where the Hamiltonian is

$$\begin{aligned} \mathcal{H}^{(0)} &= \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{2}A_\mu A^\mu - \frac{1}{m}\epsilon_{ijk}B^{0i}\partial^j A^k \\ &\quad - \frac{1}{2m}A^0\epsilon_{ijk}\partial^i B^{jk}. \end{aligned} \quad (2.8)$$

Here, we adopt the convention: $\epsilon_{0ijk} = \epsilon_{ijk}$, $\epsilon^{123} = +1$, and $g^{\mu\nu} = (+---)$. As is clear in the Lagrangian (2.7), there is no need to introduce additional auxiliary fields such as momenta because it already has the form of the first order.

Then, we identify the initial sets of symplectic variables and their conjugate momenta from the first order Lagrangian as follows:

$$\begin{aligned} \xi^{(0)\alpha} &= (A^i, B^{ij}, A^0, B^{0i}), \\ a_\alpha^{(0)} &= \left(\frac{1}{4m}\epsilon_{ijk}B^{jk}, -\frac{1}{2m}\epsilon_{ijk}A^k, 0, \mathbf{0}^T \right). \end{aligned} \quad (2.9)$$

Note that the coefficients of the fields having a time derivative in the canonical sector play the roles of momenta, and the B^{ij} in the symplectic variable $\xi^{(0)\alpha}$ denote the independent component fields such as $B^{ij} := (B^{12}, B^{23}, B^{31})$ in order, collectively, while any contracted indices are understood to be summed over unless otherwise mentioned. From the definition of the symplectic two-form matrix [9]:

$$f_{\alpha\beta}(x, y) = \frac{\partial a_\beta(y)}{\partial \xi^\alpha(x)} - \frac{\partial a_\alpha(x)}{\partial \xi^\beta(y)}, \quad (2.10)$$

we explicitly obtain the following zeroth-iterated matrix:

$$f_{\alpha\beta}^{(0)}(x, y) = \begin{pmatrix} O & \frac{1}{m}\epsilon_{i(jk)} \mathbf{0} & O \\ -\frac{1}{m}\epsilon_{(ij)k} & O & \mathbf{0} & O \\ \mathbf{0}^T & \mathbf{0}^T & 0 & \mathbf{0}^T \\ O & O & \mathbf{0} & O \end{pmatrix} \delta(x-y). \quad (2.11)$$

The tensor components $\epsilon_{i(jk)}$, $\epsilon_{(ij)k}$ are given by

$$\epsilon_{i(jk)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (2.12)$$

where the i/jk -components outside/inside the parentheses denote explicitly the vector/tensor fields and behave like the totally antisymmetric tensor ϵ_{ijk} , similar to $\epsilon_{(ij)k}$.

Hereafter, let us omit the parentheses; otherwise confusion arises. We also note that in the matrix $f_{\alpha\beta}^{(0)}$ the O , $\mathbf{0}$ and $\mathbf{0}^T$ symbols stand for a 3×3 null matrix, a three-dimensional column null vector, and its transpose, respectively. Since we easily can see that the matrix $f_{\alpha\beta}^{(0)}$ is singular, there exist four-fold infinities related to zero modes, $\tilde{\nu}_\alpha^{(0)T}(\sigma, x)$, labelled by discrete indices $\sigma = (\epsilon_1, \epsilon_2)$ with $\epsilon_2 \equiv ((\epsilon_2^1, 0, 0), (0, \epsilon_2^2, 0), (0, 0, \epsilon_2^3))$, where ϵ_1 and ϵ_2 are arbitrary functions of the continuum label x , explicitly with components

$$\begin{aligned}\tilde{\nu}_\alpha^{(0)T}(\epsilon_1, x) &= (\mathbf{0}, \mathbf{0}, \epsilon_1, \mathbf{0}), \\ \tilde{\nu}_\alpha^{(0)T}(\epsilon_2, x) &= (\mathbf{0}, \mathbf{0}, 0, \epsilon_2).\end{aligned}\quad (2.13)$$

Therefore, the four-fold zero modes $\tilde{\nu}_\alpha^{(0)T}(\sigma, x)$ generate four corresponding Lagrangian constraints Ω_{ϵ_1} and Ω_{ϵ_2} defined by

$$\begin{aligned}\int dx \Omega_\sigma(x) &= \int dx \tilde{\nu}_\alpha^{(0)T}(\sigma, x) \frac{\partial}{\partial \xi^{(0)\alpha}(x)} \int dy \mathcal{H}^{(0)}(y) \\ &= 0,\end{aligned}$$

such that

$$\begin{aligned}\Omega_{\epsilon_1} &= A_0 + \frac{1}{2m} \epsilon_{ijk} \partial^i B^{jk} = 0, \\ \Omega_{\epsilon_2} &= B_{0i} - \frac{1}{m} \epsilon_{ijk} \partial^j A^k = 0.\end{aligned}\quad (2.14)$$

These constraints should be conserved in time, which requirement is incorporated into the Lagrangian (2.7), resulting in an extension of the symplectic space with auxiliary fields, α , β^i , which correspond to Lagrange multipliers.

As a result, the first-iterated Lagrangian is written

$$\mathcal{L}^{(1)} = \frac{1}{4m} \epsilon_{ijk} \dot{A}^i B^{jk} - \frac{1}{4m} \epsilon_{ijk} \dot{B}^{ij} A^k + \Omega_{\epsilon_1} \dot{\alpha} + \Omega_{\epsilon_2} \dot{\beta}^i - \mathcal{H}^{(1)}, \quad (2.15)$$

where the first-iterated Hamiltonian is given by

$$\begin{aligned}\mathcal{H}^{(1)}(\xi) &= \mathcal{H}^{(0)}|_{\Omega_\sigma=0} \\ &= \frac{1}{4} B_{ij} B^{ij} - \frac{1}{2} A_i A^i - \frac{1}{2} B_{0i} B^{0i} + \frac{1}{2} A_0 A^0.\end{aligned}\quad (2.16)$$

We have now for the new symplectic variables and their conjugate momenta

$$\begin{aligned}\xi^{(1)\alpha} &= \{A^i, B^{ij}, A^0, B^{0i}, \alpha, \beta^i\}, \\ a_\alpha^{(1)} &= \left\{ \frac{1}{4m} \epsilon_{ijk} B^{jk}, -\frac{1}{2m} \epsilon_{ijk} A^k, 0, \mathbf{0}^T, \Omega_{\epsilon_1}, \Omega_{\epsilon_2} \right\}.\end{aligned}\quad (2.17)$$

The first-iterated symplectic matrix is again obtained:

$$f_{\alpha\beta}^{(1)}(x, y) = \begin{pmatrix} O & \frac{1}{m} \epsilon_{ijk} & \mathbf{0} & O & \mathbf{0} & \frac{1}{m} \epsilon_{ijk} \partial^k \\ -\frac{1}{m} \epsilon_{ijk} & O & \mathbf{0} & O & -\frac{1}{m} \epsilon_{ijk} \partial^k & O \\ \mathbf{0}^T & \mathbf{0}^T & 0 & \mathbf{0}^T & 1 & \mathbf{0}^T \\ O & O & \mathbf{0} & O & \mathbf{0} & -\delta_{ij} \\ \mathbf{0}^T & -\frac{1}{m} \epsilon_{ijk} \partial^k & -1 & \mathbf{0}^T & 0 & \mathbf{0}^T \\ \frac{1}{m} \epsilon_{ijk} \partial^k & O & \mathbf{0} & \delta_{ij} & \mathbf{0} & O \end{pmatrix}$$

$$\times \delta(x - y). \quad (2.18)$$

One can now easily see that this matrix has an inverse:

$$\begin{aligned}(f_{\alpha\beta}^{(1)})^{-1}(x, y) &= \\ &= \begin{pmatrix} O & -m\epsilon_{ijk} \partial^i & O & \mathbf{0} & O & O \\ m\epsilon_{ijk} & O & \mathbf{0} & F_{24} & \mathbf{0} & O \\ \partial^i & \mathbf{0}^T & 0 & \mathbf{0}^T & -1 & \mathbf{0}^T \\ O & F_{24}^T & \mathbf{0} & O & \mathbf{0} & \delta_{ij} \\ \mathbf{0}^T & \mathbf{0}^T & 1 & \mathbf{0}^T & 0 & \mathbf{0}^T \\ O & O & \mathbf{0} & -\delta_{ij} & \mathbf{0} & O \end{pmatrix} \delta(x - y),\end{aligned}\quad (2.19)$$

where

$$F_{24} = \begin{pmatrix} \partial^2 & -\partial^1 & 0 \\ 0 & \partial^3 & -\partial^2 \\ -\partial^3 & 0 & \partial^1 \end{pmatrix}. \quad (2.20)$$

Since there are no more new non-trivial zero modes, the iteration process has stopped at this stage. From this inverse matrix we easily read the desired non-vanishing symplectic brackets for the vector and the tensor fields

$$\begin{aligned}\{A^0(x), A^i(y)\} &= \partial_x^i \delta(x - y), \\ \{A^i(x), B^{jk}(y)\} &= m\epsilon^{ijk} \delta(x - y), \\ \{B^{0i}(x), B^{jk}(y)\} &= (\delta^{ij} \partial_x^k - \delta^{ik} \partial_x^j) \delta(x - y).\end{aligned}\quad (2.21)$$

This ends the usual symplectic procedure with only the four true constraints in (2.14).

Comparing this symplectic scheme with the Dirac formulation, one easily sees that the former has a smaller number of constraints than that of the latter which has 14 constraints as shown in (4.2) and (4.3) in Sect. 4, one does not need to define primary constraints, and thus it is more efficient to get the brackets. In this respect, it is generally understood that the symplectic method deals only with true constraints while Dirac's one is over-constrained.

3 Gauge invariant symplectic embedding

In this section, we embed the first order master Lagrangian without resorting to the Hamiltonian method, which is the usual way of getting the corresponding gauge invariant Lagrangian. In the usual BFT embedding scheme, one first works with the full Dirac constraints and the Hamiltonian defined in the phase space, converts them into gauge invariant ones through the systematic BFT scheme in an extended phase space, and finally using the path integral methods performs a series of integrations for the momentum variables to find out a corresponding gauge invariant Lagrangian in an extended configuration space. Compared with this rather long procedure, the gauge invariant symplectic embedding scheme has the merit of simplicity based on the singular property of the symplectic matrix with corresponding ‘‘trivial’’ zero modes which will be defined shortly.

The idea is simply to consider that the desired gauge invariant Lagrangian resulting from an embedding procedure would be provided by

$$\mathcal{L}_T = \mathcal{L}_O + \mathcal{L}_{WZ}, \quad (3.1)$$

where the Lagrangian \mathcal{L}_O is the symmetrized original master Lagrangian $\mathcal{L}^{(0)}$ in (2.7). Then the symplectic procedure is greatly simplified, if we make the following “educated” guess for WZ Lagrangian², respecting the Lorentz symmetry:

$$\mathcal{L}_{WZ} = c_1 \partial_\mu \theta \partial^\mu \theta + c_2 A^\mu \partial_\mu \theta + c_3 Q_{\mu\nu} Q^{\mu\nu} + c_4 B_{\mu\nu} Q^{\mu\nu}, \quad (3.2)$$

where $Q_{\mu\nu} = \partial_\mu Q_\nu - \partial_\nu Q_\mu$. Here θ and Q^μ are gauge degrees of freedom with respect to the original fields A^μ and $B^{\mu\nu}$, respectively. As an Ansatz for a consistent Lorentz covariant embedding, we shall take the coefficients c_1, c_2, c_3, c_4 to be constants, and we will fix these by the condition that a finally-iterated symplectic matrix has “trivial” zero modes on the one hand and these zero modes should generate no new constraints on the other hand. Furthermore, we will show that the “trivial” zero modes related with the finally-iterated symplectic matrix correctly yield the proper rules of gauge transformation including a first-stage reducible trivial zero mode associated with a reducible constraint.

Now we could read the canonical momenta of the total Lagrangian

$$\begin{aligned} \pi_0 &= -c_2 \theta, & \pi_\theta &= 2c_1 \dot{\theta}, \\ \pi_{0i} &= -2c_4 Q_i, & P_i &= 4c_3 (\dot{Q}_i - \partial_i Q_0). \end{aligned} \quad (3.3)$$

Here we have used the partially integrated Lagrangian for the second and fourth terms in (3.2) in order to easily show the coincidence with the constraints obtained from the BFT embedding in Sect. 4. Along with these auxiliary variables (momenta), to implement the symplectic procedure as in the previous section, we write the total Lagrangian (3.1) in the first ordered form:

$$\begin{aligned} \mathcal{L}_T^{(0)} &= \frac{1}{4m} \epsilon_{ijk} B^{jk} \dot{A}^i - \frac{1}{4m} \epsilon_{ijk} A^k \dot{B}^{ij} + \pi_0 \dot{A}^0 + \pi_\theta \dot{\theta} \\ &\quad + \pi_{0i} \dot{B}^{0i} + P_i \dot{Q}^i - \mathcal{H}_T^{(0)}, \end{aligned} \quad (3.4)$$

where the Hamiltonian is given by

$$\mathcal{H}_T^{(0)} = \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{2} A_\mu A^\mu - \frac{1}{m} \epsilon_{ijk} B^{0i} \partial^j A^k$$

² Similar to the Proca model [14], one could consider the Ansatz more general and take manifestly Lorentz invariant functions of the WZ Lagrangian composed of a scalar θ and vector fields Q^μ ; however, this consideration only adds an unnecessary calculational complication. On the other hand, in recent works [15], there is made an attempt to make the Lagrangian embedding a bit systematic; however, in practice it seems very complicated even in the simplest one-form cases [12–14] including the Proca model. Furthermore, it may be intractable to apply the method to reducible systems, including the massive vector–tensor theory with topological coupling.

$$\begin{aligned} & - \frac{1}{2m} A^0 \epsilon_{ijk} \partial^i B^{jk} + \frac{1}{4c_1} \pi_\theta^2 + \frac{1}{8c_3} P_i P^i \\ & - c_1 \partial_i \theta \partial^i \theta - c_2 A^i \partial_i \theta - c_3 Q_{ij} Q^{ij} - c_4 B_{ij} Q^{ij} \\ & + P_i \partial^i Q^0 + 2c_4 B^{0i} \partial_i Q_0. \end{aligned} \quad (3.5)$$

Then, the initial set of symplectic variables $\xi^{(0)\alpha}$ and their conjugate momenta $a_\alpha^{(0)}$ are given by

$$\begin{aligned} \xi^{(0)\alpha} &= (A^i, B^{ij}, \theta, \pi_\theta, Q^i, P_i, A^0, B^{0i}, Q^0), \\ a_\alpha^{(0)} &= \left(\frac{1}{4m} \epsilon_{ijk} B^{jk}, -\frac{1}{2m} \epsilon_{ijk} A^k, \pi_\theta, 0, P_i, \mathbf{0}^T, -c_2 \theta, -2c_4 Q_i, 0 \right). \end{aligned} \quad (3.6)$$

From the above set of the symplectic variables we read off the symplectic matrix defined in (2.10) to be

$$\begin{aligned} f_{\alpha\beta}^{(0)}(x, y) &= \begin{pmatrix} O & \frac{1}{m} \epsilon_{ijk} & \mathbf{0} & \mathbf{0} & O & O & \mathbf{0} & O & \mathbf{0} \\ -\frac{1}{m} \epsilon_{ijk} & O & \mathbf{0} & \mathbf{0} & O & O & \mathbf{0} & O & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & 0 & -1 & \mathbf{0}^T & \mathbf{0}^T & -c_2 & \mathbf{0}^T & 0 \\ \mathbf{0}^T & \mathbf{0}^T & 1 & 0 & \mathbf{0}^T & \mathbf{0}^T & 0 & \mathbf{0}^T & 0 \\ O & O & \mathbf{0} & \mathbf{0} & O & -\delta_{ij} & 0 & 2c_4 \delta_{ij} & \mathbf{0} \\ O & O & \mathbf{0} & \mathbf{0} & \delta_{ij} & O & \mathbf{0} & O & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & c_2 & 0 & \mathbf{0}^T & \mathbf{0}^T & 0 & \mathbf{0}^T & 0 \\ O & O & \mathbf{0} & \mathbf{0} & -2c_4 \delta_{ij} & O & \mathbf{0} & O & \mathbf{0} \\ O & O & \mathbf{0} & \mathbf{0} & O & O & \mathbf{0} & O & \mathbf{0} \end{pmatrix} \\ &\quad \times \delta(x - y), \end{aligned} \quad (3.7)$$

which is manifestly singular as we observe the last null row and column, and thus the matrix has non-trivial zero modes given by

$$\begin{aligned} \nu_\alpha^{(0)T}(\epsilon_1, x) &= (\mathbf{0}^T, \mathbf{0}^T, 0, -c_2 \epsilon_1, \mathbf{0}^T, \mathbf{0}^T, \epsilon_1, \mathbf{0}^T, 0), \\ \nu_\alpha^{(0)T}(\epsilon_2, x) &= (\mathbf{0}^T, \mathbf{0}^T, 0, 0, \mathbf{0}^T, 2c_4 \epsilon_2, 0, \epsilon_2, 0), \\ \nu_\alpha^{(0)T}(\epsilon_3, x) &= (\mathbf{0}^T, \mathbf{0}^T, 0, 0, \mathbf{0}^T, \mathbf{0}^T, 0, \mathbf{0}^T, \epsilon_3). \end{aligned} \quad (3.8)$$

Applying these zero modes from the left to the equation of motion we have obtained the constraints $\phi_\sigma \equiv (\phi_{\epsilon_1}, \phi_{\epsilon_2}, \phi_{\epsilon_3})$:

$$\begin{aligned} \phi_{\epsilon_1} &= A_0 + \frac{1}{2m} \epsilon_{ijk} \partial^i B^{jk} + \frac{c_2}{2c_1} \pi_\theta = 0, \\ \phi_{\epsilon_2} &= B_{0i} - \frac{1}{m} \epsilon_{ijk} \partial^j A^k - \frac{c_4}{2c_3} P_i = 0, \\ \phi_{\epsilon_3} &= \partial^i P_i + 2c_4 \partial_i B^{0i} = 0. \end{aligned} \quad (3.9)$$

Next, following the symplectic algorithm for theories having gauge symmetry [9, 12, 14], we obtain the first-iterated Lagrangian by enlarging the canonical sector with the constraints ϕ_σ and their associated Lagrangian multipliers $(\alpha, \beta^i, \gamma)$, respectively, as follows:

$$\begin{aligned} \mathcal{L}_T^{(1)} &= \frac{1}{4m} \epsilon_{ijk} B^{jk} \dot{A}^i - \frac{1}{4m} \epsilon_{ijk} A^k \dot{B}^{ij} \\ &\quad - 2c_2 \theta \dot{A}^0 + \pi_\theta \dot{\theta} - 2c_4 Q_i \dot{B}^{0i} + P_i \dot{Q}^i + \phi_{\epsilon_1} \dot{\alpha} + \phi_{\epsilon_2} \dot{\beta}^i \\ &\quad + \phi_{\epsilon_3} \dot{\gamma} - \mathcal{H}_T^{(1)}, \end{aligned} \quad (3.10)$$

where $\mathcal{H}_T^{(1)} = \mathcal{H}^{(0)}|_{\phi_{\epsilon_3}=0}$ is the first-iterated Hamiltonian. We have now for the first-level symplectic variables $\xi^{(1)\alpha}$ and their conjugate momenta $a_\alpha^{(1)}$

$$\begin{aligned} \xi^{(1)\alpha} &= (A^i, B^{ij}, \theta, \pi_\theta, Q^i, P_i, A^0, B^{0i}, \alpha, \beta^i, \gamma), \\ a_\alpha^{(1)} &= \left(\frac{1}{4m} \epsilon_{ijk} B^{jk}, -\frac{1}{2m} \epsilon_{ijk} A^k, \right. \\ &\quad \left. \pi_\theta, 0, P_i, \mathbf{0}^T, 0, \mathbf{0}^T, \phi_{\epsilon_1}, \phi_{\epsilon_2}, \phi_{\epsilon_3} \right), \end{aligned} \quad (3.11)$$

and the first-iterated symplectic matrix now reads

$$f_{\alpha\beta}^{(1)}(x, y) = \begin{pmatrix} f_{\hat{\alpha}\hat{\beta}}^{(0)} & M_{\hat{\alpha}\sigma} \\ -M_{\sigma\hat{\alpha}}^T & O \end{pmatrix} \delta(x - y), \quad (3.12)$$

where the submatrix $f_{\hat{\alpha}\hat{\beta}}^{(0)}$ refers to the $\xi_{\hat{\alpha}} = (A^i, B^{ij}, \theta, \pi_\theta, Q^i, P_i, A^0, B^{0i})$ sector, and $M_{\hat{\alpha}\sigma}$ is a 18×5 matrix defined as $M_{\hat{\alpha}\sigma} = \frac{\partial \phi_\sigma(y)}{\partial \xi_{\hat{\alpha}}(x)}$:

$$M_{\hat{\alpha}\sigma}(x, y) = \begin{pmatrix} \mathbf{0} & -\frac{1}{m} \epsilon_{ijk} \partial_y^k & \mathbf{0} \\ \frac{1}{m} \epsilon_{ijk} \partial_y^k & O & \mathbf{0} \\ 0 & \mathbf{0}^T & 0 \\ \frac{c_2}{2c_1} & \mathbf{0}^T & 0 \\ \mathbf{0} & O & \mathbf{0} \\ \mathbf{0} & -\frac{c_4}{2c_3} \delta_{ij} & \partial_y^i \\ 1 & \mathbf{0}^T & 0 \\ \mathbf{0} & -\delta_{ij} & 2c_4 \partial_i^y \end{pmatrix} \delta(x - y). \quad (3.13)$$

Then this symplectic matrix, which contains no null rows and columns, is seemingly non-singular. Note that we observe that the Q^0 term does not appear in the canonical sector of the Lagrangian $\mathcal{L}^{(1)}$, nor in the Hamiltonian $\mathcal{H}_T^{(1)}$ due to the use of the constraint ϕ_{ϵ_3} . It is also important to eliminate the Q^0 field in the first-iterated level in the technical point of view. Keeping the Q^0 field in the Lagrangian $\mathcal{L}^{(1)}$ leads to trouble in the symplectic matrix because there is always a zero row or column due to the absence of its conjugate momentum in $a^{(1)}(\xi)$. This prevents us from finding the desired zero modes.

Now, the essential point of the symplectic Lagrangian embedding [12–15] is this: in order to realize a gauge symmetry in this approach, the matrix (3.12) must have at least one “trivial” zero mode which does not generate a new constraint. In our case, the solution of

$$\int dy f_{\alpha\beta}^{(1)}(x, y) \nu_\beta^{(1)}(y) = 0$$

yields the following “trivial” zero modes:

$$\begin{aligned} \nu_\alpha^{(1)\text{T}}(\epsilon_1, x) &= (\mathbf{0}^T, \mathbf{0}^T, 0, -c_2 \epsilon_1, \mathbf{0}^T, \mathbf{0}^T, \epsilon_1, \mathbf{0}^T, 0, \mathbf{0}^T, 0), \\ \nu_\alpha^{(1)\text{T}}(\epsilon_2, x) &= (\mathbf{0}^T, \mathbf{0}^T, 0, 0, \mathbf{0}^T, 2c_4 \epsilon_2, 0, \epsilon_2, 0, \mathbf{0}^T, 0), \\ \nu_\alpha^{(1)\text{T}}(\epsilon_3, x) &= \end{aligned}$$

$$\begin{aligned} &\left(\partial^i \epsilon_3, \mathbf{0}^T, -\frac{1}{c_2} \epsilon_3, 0, \mathbf{0}^T, \mathbf{0}^T, 0, \mathbf{0}^T, \epsilon_3, \mathbf{0}^T, 0 \right), \\ \nu_\alpha^{(1)\text{T}}(\epsilon_4, x) &= \left(\mathbf{0}^T, F_{24}^T \epsilon_4, 0, 0, -\frac{1}{2c_4} \epsilon_4, \mathbf{0}^T, 0, \mathbf{0}^T, 0, \epsilon_4, 0 \right), \\ \nu_\alpha^{(1)\text{T}}(\epsilon_5, x) &= (\mathbf{0}^T, \mathbf{0}^T, 0, 0, \partial_i \epsilon_5, \mathbf{0}^T, 0, \mathbf{0}^T, 0, \mathbf{0}^T, \epsilon_5), \end{aligned} \quad (3.14)$$

where F_{24}^T is defined in (2.20), while giving the relations for the free adjustable coefficients:

$$c_2^2 = 2c_1, \quad c_4^2 = -c_3, \quad (3.15)$$

which are obtained from the single-valuedness of the zero mode functions.

For consistency, we can confirm that these zero modes do not really generate any new constraint provided we apply them to the right-hand side of the equations of motion,

$$\begin{aligned} \int dx \nu_\alpha^{(1)}(\sigma, x) \frac{\partial}{\partial \xi^{(1)\alpha}(x)} \int dy \mathcal{H}_T^{(1)}(y) &= 0, \\ \sigma &= (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5), \end{aligned}$$

explicitly:

$$\begin{aligned} &\int dx \nu_\alpha^{(1)\text{T}}(\epsilon_1, x) \frac{\delta}{\delta \xi^{(1)\alpha}(x)} \int dy \mathcal{H}^{(1)}(y) \\ &= - \int dx \epsilon_1 \phi_{\epsilon_1} = 0, \\ &\int dx \nu_\alpha^{(1)\text{T}}(\epsilon_2, x) \frac{\delta}{\delta \xi^{(1)\alpha}(x)} \int dy \mathcal{H}^{(1)}(y) \\ &= \int dx \epsilon_2 \phi_{\epsilon_2} = 0, \\ &\int dx \nu_\alpha^{(1)\text{T}}(\epsilon_3, x) \frac{\delta}{\delta \xi^{(1)\alpha}(x)} \int dy \mathcal{H}^{(1)}(y) \\ &= \int dx \epsilon_3 \left(1 - \frac{c_2^2}{2c_1} \right) \partial_i A^i = 0, \\ &\int dx \nu_\alpha^{(1)\text{T}}(\epsilon_4, x) \frac{\delta}{\delta \xi^{(1)\alpha}(x)} \int dy \mathcal{H}^{(1)}(y) \\ &= - \int dx \epsilon_4^i \left(1 + \frac{c_4^2}{c_3} \right) \partial^j B_{ij} = 0, \\ &\int dx \nu_\alpha^{(1)\text{T}}(\epsilon_5, x) \frac{\delta}{\delta \xi^{(1)\alpha}(x)} \int dy \mathcal{H}^{(1)}(y) = 0, \end{aligned} \quad (3.16)$$

where the third and fourth equations are reconfirmed by using the relations in (3.15). Since the last equation concerning the zero mode ϵ_5 identically vanishes, this false zero mode plays no role at all, reflecting that the symplectic scheme deals only with true constraints.

Therefore, before proceeding to the symplectic embedding algorithm, with these determinations of the coefficients we should go back to the first-iterated level (3.10) because the constraints are now not all independent, i.e.,

we can identify the last two constraints in (3.9) as follows:

$$\partial^i \phi_{\epsilon_2^i} - \frac{1}{2c_4} \phi_{\epsilon_3} = 0, \tag{3.17}$$

which means that the gauge invariant Lagrangian we are seeking is reducible. In this case, in order to resolve the reducible constraint, we have to modify the auxiliary field as $\beta^i \rightarrow \beta^i - 2c_4 \partial^i \gamma$. Then, a pair of the symplectic variable and the momentum, $(\gamma, \phi_{\epsilon_3})$, are absorbed into the modified β^i variables. This also modifies the symplectic matrix (3.12) to a new one which does not have the last row and column like the one in (3.7). Finding solutions having new zero modes is exactly equivalent in the corresponding “trivial” zero modes (3.16) to set the ϵ_5 -parameter zero. This clearly explains why the last zero mode in (3.14) does not generate any new constraints but vanishes identically.

On the other hand, we could also interpret these results in view of the linear dependence of the “trivial” zero modes as follows. In the zero mode solutions (3.14) we have explicitly inserted the gauge parameters ϵ_σ keeping the form of the gauge transformations (3.21) in mind. Instead, since we could normalize them by introducing the delta functional³ in the symplectic embedding scheme [14], we then have the last two zero modes in (3.14), explicitly:

$$\begin{aligned} & \nu_{\hat{\alpha},y}^{(1)\text{T}}(\epsilon_4, x) \\ &= \left(\mathbf{0}^{\text{T}}, F_{24}^{\text{T}}, 0, 0, -\frac{1}{2c_4} \mathbf{e}, \mathbf{0}^{\text{T}}, 0, \mathbf{0}^{\text{T}} \right) \delta(x-y), \\ & \nu_{\hat{\alpha},y}^{(1)\text{T}}(\epsilon_5, x) \\ &= \left(\mathbf{0}^{\text{T}}, \mathbf{0}^{\text{T}}, 0, 0, \nabla \epsilon_5, \mathbf{0}^{\text{T}}, 0, \mathbf{0}^{\text{T}} \right) \delta(x-y), \end{aligned} \tag{3.18}$$

where the $\hat{\alpha}$ denote the component fields $\xi_{\hat{\alpha}} = (A^i, B^{ij}, \theta, \pi_\theta, Q^i, P_i, A^0, B^{0i})$ and \mathbf{e} is a unit 3-dimensional vector. Then, we easily see that the zero mode concerning the gauge parameter ϵ_5 is related to $\nu_{\hat{\alpha},y}^{(1)\text{T}}(\epsilon_4, x)$ by

$$\nu_{\hat{\alpha},y}^{(1)\text{T}}(\epsilon_5, x) = -2c_4 \nabla \nu_{\hat{\alpha},y}^{(1)\text{T}}(\epsilon_4, x) \tag{3.19}$$

showing that it is not linearly independent, i.e., reducible. From this point of view, we would call it, $\nu_{\hat{\alpha},y}^{(1)\text{T}}(\epsilon_5, x)$, a “first-stage reducible” trivial zero mode. This extends previously known results [12–15] in which the existence of “trivial” zero modes, or equivalently no new constraints are generated in the symplectic framework, implies gauge symmetry in the system to include the first-stage reducible case.

As a result, we arrive at the final result on the symplectic embedding of the massive vector–tensor theory with topological coupling. The desired gauge invariant Lagrangian is now explicitly written

$$\mathcal{L}_T = \mathcal{L}_O + \mathcal{L}_{WZ}$$

³ We can also label the zero modes as follows: $\nu_{\hat{\alpha},y}^{(l)}(\sigma, x)$ ($\sigma = 1, \dots, N$), where “ l ” refers to the “level”, $\hat{\alpha}, y$ stand for the component, while σ, x label the N -fold infinity of the zero modes in \mathcal{R}^3 . See for more details [14].

$$\begin{aligned} &= -\frac{1}{4} (B_{\mu\nu} - 2c_4 Q_{\mu\nu})^2 + \frac{1}{2} (A_\mu + c_2 \partial_\mu \theta)^2 \\ &+ \frac{1}{2m} \epsilon_{\mu\nu\rho\sigma} B^{\mu\nu} \partial^\rho A^\sigma. \end{aligned} \tag{3.20}$$

This ends the Lagrangian embedding in the symplectic setup.

Now, in order to discuss the gauge transformation, we consider the symplectic matrix (3.12). As has been shown in [9, 12], the “trivial” zero modes generate gauge transformations on the symplectic variable $\xi_{\hat{\alpha}}^{(1)}$ in the sense that

$$\delta \xi_{\hat{\alpha}}(x) = \Sigma_{\tilde{\sigma}} \nu_{\hat{\alpha}}^{(1)\text{T}}(\tilde{\sigma}, x), \quad \tilde{\sigma} = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4). \tag{3.21}$$

We thus obtain the gauge transformations of the symplectic variables from (3.14) as follows:

$$\begin{aligned} \delta A^0 &= \epsilon_1, \quad \delta A^i = \partial^i \epsilon_3, \quad \delta \theta = -\frac{1}{c_2} \epsilon_3, \\ \delta B^{0i} &= \epsilon_2^i, \quad \delta B^{ij} = -(\partial^i \epsilon_4^j - \partial^j \epsilon_4^i), \quad \delta Q^i = -\frac{1}{2c_4} \epsilon_4^i, \end{aligned} \tag{3.22}$$

except the missing transformation δQ^0 which cannot be obtained at this stage due to the elimination of the Q^0 component in the first-iterated level of the procedure. This is very similar to the BFT Hamiltonian embedding for the constrained reducible system in which we should introduce the Q^0 field in the path integral measure in order to keep manifest covariance in the Lagrangian⁴.

To find out explicitly the gauge transformation for the Q^0 field as well as the existence of possible restrictions to the gauge parameters, let us consider the total variation of the Lagrangian (3.20), which transforms as

$$\begin{aligned} \delta \mathcal{L}_T &= (A_0 + c_2 \partial_0 \theta)(\epsilon_1 - \partial^0 \epsilon_3) \\ &+ \left(\frac{1}{m} \epsilon_{ijk} \partial^j A^k - B_{0i} + 2c_4 Q_{0i} \right) \\ &\times (\epsilon_2^i + \partial^0 \epsilon_4^i + 2c_4 \partial^i \delta Q^0). \end{aligned} \tag{3.23}$$

Therefore, if we identify the gauge parameters as

$$\epsilon_1 = \partial^0 \epsilon_3, \quad \epsilon_2^i = -(\partial^0 \epsilon_4^i - \partial^i \epsilon_4^0), \tag{3.24}$$

along with a new parameter ϵ_4^0 defined by

$$\delta Q^0 = -\frac{1}{2c_4} \epsilon_4^0, \tag{3.25}$$

we can make the total Lagrangian \mathcal{L}_T gauge invariant. As a result, this Lagrangian is invariant under the final gauge transformations:

$$\begin{aligned} \delta A^\mu &= \partial^\mu \epsilon_3, & \delta \theta &= -\frac{1}{c_2} \epsilon_3, \\ \delta B^{\mu\nu} &= -(\partial^\mu \epsilon_4^\nu - \partial^\nu \epsilon_4^\mu), & \delta Q^\mu &= -\frac{1}{2c_4} \epsilon_4^\mu. \end{aligned} \tag{3.26}$$

⁴ We will discuss this explicitly in the next section.

Therefore, by considering purely the symplectic Lagrangian embedding, we have found the gauge invariant Lagrangian as well as their full gauge transformations including δQ^0 . Note that the coefficients c_2, c_4 can be either rescaled on the variables as $(\theta, Q^\mu) \rightarrow (c_2\theta, 2c_4Q^\mu)$, or fixed as $c_2 = \pm 1, c_4 = \pm \frac{1}{2}$. Then the resulting Lagrangian is reduced to the well-known gauge embedded form of the massive vector–tensor theory with topological coupling,

$$\begin{aligned} \mathcal{L}_T = & -\frac{1}{4}(B_{\mu\nu} - Q_{\mu\nu})^2 + \frac{1}{2}(A_\mu + \partial_\mu\theta)^2 \\ & + \frac{1}{2m}\epsilon_{\mu\nu\rho\sigma}B^{\mu\nu}\partial^\rho A^\sigma, \end{aligned} \quad (3.27)$$

where we set the free adjustable coefficients $c_2 = 1$ and $c_4 = 1/2$, and this Lagrangian is invariant under

$$\begin{aligned} \delta A^\mu &= \partial^\mu\epsilon, & \delta\theta &= -\epsilon, \\ \delta B^{\mu\nu} &= \partial^\mu\epsilon^\nu - \partial^\nu\epsilon^\mu, & \delta Q^\mu &= \epsilon^\mu, \end{aligned} \quad (3.28)$$

where the gauge parameters are redefined by $\epsilon = \epsilon_3$ and $\epsilon^\mu = -\epsilon_4^\mu$.

4 Revisited BFT embedding

In this section, we will compare the symplectic Lagrangian embedding with the previous work [7] of the BFT Hamiltonian one, which makes a second-class constraint Hamiltonian system into the corresponding first-class one in a systematic way.

First, let us start with the canonical momenta from the symmetrized form of the Lagrangian (2.1) with $\pi_0 = 0$, $\pi_i = \frac{1}{4m}\epsilon_{ijk}B^{jk}$, $\pi_{0i} = 0$, and $\pi_{ij} = -\frac{1}{2m}\epsilon_{ijk}A^k$ in order to analyze the Hamiltonian structure of the model. Then we have obtained the primary Hamiltonian in Dirac's terminology:

$$\begin{aligned} \mathcal{H}_p = & \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{2}A_\mu A^\mu - \frac{1}{m}\epsilon^{ijk}B_{0i}\partial_j A_k \\ & - \frac{1}{2m}A^0\epsilon^{ijk}\partial_i B_{jk} + \lambda^0\pi_0 + \lambda^i\Omega_i + \Sigma^{0i}\pi_{0i} + \Sigma^{ij}\Omega_{ij}, \end{aligned} \quad (4.1)$$

where the ten primary constraints are given by

$$\begin{aligned} \pi_0 \approx 0, & \quad \Omega_i \equiv \pi_i - \frac{1}{4m}\epsilon_{ijk}B^{jk} \approx 0, \\ \pi_{0i} \approx 0, & \quad \Omega_{ij} \equiv \pi_{ij} + \frac{1}{2m}\epsilon_{ijk}A^k \approx 0, \end{aligned} \quad (4.2)$$

along with their associated Lagrange multipliers $\lambda^0, \lambda^i, \Sigma^{0i}$, and Σ^{ij} .

There are four additional secondary constraints which are obtained from the time stability conditions of the primary constraints π_0 and π_{0i} , thus:

$$\begin{aligned} \Lambda &\equiv A_0 + \frac{1}{2m}\epsilon_{ijk}\partial^i B^{jk} \approx 0, \\ \Lambda_i &\equiv B_{0i} - \frac{1}{m}\epsilon_{ijk}\partial^j A^k \approx 0. \end{aligned} \quad (4.3)$$

As expected, the Lagrange multipliers Σ^{ij}, λ^i corresponding to the constraints Ω_i, Ω_{ij} are fixed under the time stability condition, and the other ones, λ^0, Σ^{0i} , are determined by the consistency requirement of the secondary constraints Λ, Λ_i , respectively.

Then the full set of these constraints constitutes the constraint algebra of second class as follows:

$$\begin{aligned} \{\pi_0, \Lambda\} &= -\delta(x-y), \\ \{\pi_{0i}, \Lambda_j\} &= -\delta_{ij}\delta(x-y), \\ \{\Omega_i, \Omega_{jk}\} &= -\frac{1}{m}\epsilon_{ijk}\delta(x-y), \end{aligned} \quad (4.4)$$

where we have redefined the secondary constraints by $\Lambda + \partial^i\Omega_i \rightarrow \Lambda = \partial^i\pi_i + A^0 + \frac{1}{4m}\epsilon^{ijk}\partial_i B_{jk} \approx 0$, and $\Lambda_i + \partial^j\Omega_{ij} \rightarrow \Lambda_i = \partial^j\pi_{ij} + B_{0i} - \frac{1}{2m}\epsilon_{ijk}\partial^j A^k \approx 0$. These new definitions make the constraint algebra have no derivative terms. As a result, we have easily obtained the following Dirac brackets:

$$\begin{aligned} \{A^0(x), A^i(y)\}_D &= \partial_x^i\delta(x-y), \\ \{A^i(x), B^{jk}(y)\}_D &= -m\epsilon^{ijk}\delta(x-y), \\ \{B^{0i}(x), B^{jk}(y)\}_D &= (\delta^{ij}\partial_x^k - \delta^{ik}\partial_x^j)\delta(x-y), \\ \{A^i(x), \pi_j(y)\}_D &= \frac{1}{2}\delta_j^i\delta(x-y), \\ \{A^0(x), \pi_{ij}(y)\}_D &= -\frac{1}{2m}\epsilon_{ijk}\partial_x^k\delta(x-y), \\ \{\pi_i(x), B_{0j}(y)\}_D &= \frac{1}{2m}\epsilon_{ijk}\partial_x^k\delta(x-y), \\ \{\pi_i(x), \pi_{jk}(y)\}_D &= \frac{1}{4m}\epsilon_{ijk}\delta(x-y), \\ \{B^{ij}(x), \pi_{kl}(y)\}_D &= \frac{1}{2}(\delta_k^i\delta_l^j - \delta_k^j\delta_l^i)\delta(x-y) \end{aligned} \quad (4.5)$$

for the massive vector–tensor theory with topological coupling.

Now, let us briefly recapitulate the BFT Hamiltonian embedding for this theory. For that purpose, we have introduced auxiliary fields having involutive relations in which not only modified new constraints in the enlarged space are strongly vanishing with each other, but also they have the Poisson brackets vanishing, not the Dirac brackets, with physical quantities such as Hamiltonian and the fields themselves. Through this BFT prescription, after introducing auxiliary fields paired as $(\theta, \pi_\theta), (Q^i, P_i)$, and (Φ^i, Φ^{jk}) , we have obtained the strongly involutive primary

$$\begin{aligned} \tilde{\pi}_0 &= \pi_0 + \theta, & \tilde{\pi}_{0i} &= \pi_{0i} + Q_i, \\ \tilde{\Omega}_i &= \Omega_i + \Phi_i, & \tilde{\Omega}_{ij} &= \Omega_{ij} + \frac{1}{m}\Phi_{ij}, \end{aligned} \quad (4.6)$$

and the secondary constraints

$$\begin{aligned} \tilde{\Lambda} &= \Lambda + \pi_\theta = A_0 + \frac{1}{2m}\epsilon_{ijk}\partial^i B^{jk} + \pi_\theta, \\ \tilde{\Lambda}_i &= \Lambda_i + P_i = B_{0i} - \frac{1}{m}\epsilon_{ijk}\partial^j A^k + P_i. \end{aligned} \quad (4.7)$$

Moreover, the strongly involutive physical fields are also obtained:

$$\begin{aligned}
\tilde{A}^0 &= A^0 + \pi_\theta, & \tilde{A}^i &= A^i + \partial^i \theta - \frac{1}{2} \epsilon^{ijk} \Phi_{jk}, \\
\tilde{\pi}_0 &= \pi_0 + \theta, & \tilde{\pi}_i &= \pi_i - \frac{1}{2m} \epsilon_{ijk} \partial^j Q^k + \frac{1}{2} \Phi_i, \\
\tilde{B}^{0i} &= B^{0i} + P^i, & \tilde{B}^{ij} &= B^{ij} - (\partial^i Q^j - \partial^j Q^i) + m \epsilon^{ijk} \Phi_k, \\
\tilde{\pi}_{0i} &= \pi_{0i} + Q_i, & \tilde{\pi}_{ij} &= \pi_{ij} - \frac{1}{2m} \epsilon_{ijk} \partial^k \theta + \frac{1}{2m} \Phi_{ij}.
\end{aligned} \tag{4.8}$$

Note that all these physical fields are terminated in the first order of the auxiliary fields, and the Poisson brackets of these fields in the extended phase space are exactly the same as the Dirac brackets (4.5) in the original phase space.

On the other hand, by using the strongly involutive physical fields (4.8), we have also obtained the extended canonical Hamiltonian:

$$\begin{aligned}
\tilde{\mathcal{H}}_c &= \frac{1}{4} (B_{ij} - Q_{ij})^2 + \frac{1}{2} m \epsilon_{ijk} (B^{ij} - Q^{ij}) \Phi^k - \frac{1}{2} m^2 \Phi_i \Phi^i \\
&\quad - \frac{1}{2} (A_i + \partial_i \theta)^2 + \frac{1}{2} \epsilon_{ijk} (A^i + \partial^i \theta) \Phi^{jk} + \frac{1}{4} \Phi_{ij} \Phi^{ij} \\
&\quad + \frac{1}{2} (B_{0i} + P_i)^2 - \frac{1}{m} \epsilon_{ijk} (B^{0i} + P^i) \partial^j A^k \\
&\quad - \frac{1}{m} (B_{0i} + P_i) \partial_j \Phi^{ij} - \frac{1}{2} (A_0 + \pi_\theta)^2 \\
&\quad - \frac{1}{2m} (A_0 + \pi_\theta) \epsilon_{ijk} \partial^i B^{jk} + (A_0 + \pi_\theta) \partial_i \Phi^i \\
&\quad + \pi_\theta \tilde{A} + P_i \tilde{A}^i,
\end{aligned} \tag{4.9}$$

where the last two terms are added to generate Gauss' constraints corresponding to the constraints, $\tilde{\pi}_0, \tilde{\pi}_{0i}$, consistently. Then the generating functional for the extended first-class systems is given by

$$\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}A^\mu \mathcal{D}\pi_\mu \mathcal{D}B^{\mu\nu} \mathcal{D}\pi_{\mu\nu} \mathcal{D}\theta \mathcal{D}\pi_\theta \mathcal{D}Q^i \mathcal{D}P_i \mathcal{D}\Phi^i \mathcal{D}\tilde{\Phi}^{ij} \\
&\quad \times \delta(\tilde{\varphi}_\alpha) \delta(\Gamma_\beta) \det | \{ \tilde{\varphi}_\alpha, \Gamma_\beta \} | e^{iS},
\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}
S &= \int d^4x \left[\pi_\mu \dot{A}^\mu + \pi_{0i} \dot{B}^{0i} + \frac{1}{2} \pi_{ij} \dot{B}^{ij} + \pi_\theta \dot{\theta} + P_i \dot{Q}^i \right. \\
&\quad \left. + \frac{1}{2} \epsilon_{ijk} \Phi^i \dot{\Phi}^{jk} - \tilde{\mathcal{H}}_c \right],
\end{aligned} \tag{4.11}$$

and the Γ_α are appropriate gauge fixing functions which have non-vanishing Poisson brackets with the modified first-class constraints $\tilde{\varphi}_\alpha = (\tilde{\pi}_0, \tilde{\pi}_{0i}, \tilde{A}, \tilde{A}_i, \tilde{\Omega}_i, \tilde{\Omega}_{ij})$.

Now, by performing the momentum integrations in the generating functional as usual [14], we have finally obtained the gauge invariant Lagrangian corresponding to $\tilde{\mathcal{H}}_c$:

$$\mathcal{Z} = \int \mathcal{D}A^\mu \mathcal{D}B^{\mu\nu} \mathcal{D}\theta \mathcal{D}Q^i \mathcal{D}\Phi^i \mathcal{D}\tilde{\Phi}^{ij}$$

$$\times \delta(Q^0) \delta(\Gamma_\beta) \det | \{ \tilde{\varphi}_\alpha, \Gamma_\beta \} | e^{iS_T}, \tag{4.12}$$

where

$$S_T = \int d^4x (\mathcal{L}_{\text{GE}} + \mathcal{L}_{\text{NWZ}}), \tag{4.13}$$

$$\begin{aligned}
\mathcal{L}_{\text{GE}} &= -\frac{1}{4} (B_{\mu\nu} - Q_{\mu\nu})^2 + \frac{1}{2} (A_\mu + \partial_\mu \theta)^2 \\
&\quad + \frac{1}{2m} \epsilon_{\mu\nu\rho\sigma} B^{\mu\nu} \partial^\rho A^\sigma, \\
\mathcal{L}_{\text{NWZ}} &= \frac{1}{2} \epsilon_{ijk} \Phi^i \dot{\Phi}^{jk} + \left[F_{i0} - \frac{m}{2} \epsilon_{ijk} (B^{jk} - Q^{jk}) + \frac{m^2}{2} \Phi_i \right] \Phi^i \\
&\quad - \left[\frac{1}{2} \epsilon_{ijk} (A^i + \partial^i \theta) + \frac{1}{m} \partial_k B_{0j} + \frac{1}{2m} \partial_0 B_{jk} + \frac{1}{4} \Phi_{ij} \right] \Phi^{jk},
\end{aligned} \tag{4.14}$$

where \mathcal{L}_{GE} is the gauge embedded Lagrangian and \mathcal{L}_{NWZ} is a new type of WZ Lagrangian. Note that we have introduced the delta functional of the variable Q^0 in the measure which serves to make the final gauge embedded Lagrangian \mathcal{L}_{GE} manifestly covariant.

On the other hand, the infinitesimal gauge transformations for the fields are given by

$$\begin{aligned}
\delta A^\mu &= -\partial^\mu \epsilon_A + \delta_j^\mu \epsilon_A^j, \\
\delta B^{\mu\nu} &= \partial^\mu \epsilon_B^\nu - \partial^\nu \epsilon_B^\mu + (\epsilon_B^{kl} - \epsilon_B^{lk}) \delta_k^\mu \delta_l^\nu, \\
\delta \theta &= \epsilon_A, \quad \delta Q^\mu = \epsilon_B^\mu, \\
\delta \Phi^i &= \frac{1}{m} \epsilon^{ijk} \epsilon_{jk}^B, \quad \delta \Phi^{ij} = -\epsilon^{ijk} \epsilon_k^A
\end{aligned} \tag{4.15}$$

from the generator of the gauge transformation⁵, the first-class constraints (4.6). The transformations related to the gauge symmetry are exactly the same as the ones (3.26) obtained from the symplectic Lagrangian scheme. On the other hand, the other parameters $\epsilon_A^i, \epsilon_B^i$ in (4.15) are not related with the gauge symmetry. These parameters are associated with the symplectic structure of the topological coupling, which are absent in the symplectic Lagrangian scheme. Since in the latter scheme the topological coupling term is already of first order, we have no need to introduce the auxiliary fields (momenta) which have become the gauge generators in the former scheme. As a result, the new type of WZ Lagrangian, which may lack manifest covariance, is indeed invariant under the extended transformations including these parameters. Moreover, by fixing unitary gauge conditions such as $\Phi^i = \tilde{\Phi}^{ij} = 0$, the new type of the WZ Lagrangian identically vanishes, while the strongly involutive constraints \tilde{A}, \tilde{A}^i in (4.6) become exactly the same as the true ones $\phi_{\epsilon_1}, \phi_{\epsilon_2}$ in (3.9) generated from the symplectic embedding procedure, respectively.

It is also important to note that in the generating functional (4.12) there exists the delta functional of the variable Q^0 which transforms as $\delta Q^0 = \epsilon_B^0$. We have introduced this

⁵ See [7] for details.

new field to make the final Lagrangian manifestly covariant. Even without this Q^0 field, we can show that the resulting Lagrangian successfully reproduces the whole BFT embedded constraint structure. However, in that case, it will lose manifest covariance. Indeed, we have introduced the new variable Q^0 in order to keep manifest covariance, while giving up the irreducible property between the constraints [7].

5 Conclusion

In this paper we have quantized a massive vector–tensor theory with topological coupling which is the first order master action of the Proca and the Kalb–Ramond models. We have adopted the symplectic scheme since it provides a relatively simple way of getting Dirac brackets rather than obtaining these brackets in the Dirac formalism for the Lagrangians having seemingly complicated antisymmetric tensor fields. In particular, we have shown that our model has only four constraints in the symplectic scheme in contrast to the Dirac (or BFT) formalism, having fourteen constraints. Moreover, we have demonstrated how the BFT embedding scheme of a second-class system into a first-class one can be realized in the framework of the symplectic approach to constrained systems. Rather than proceeding iteratively as in the BFT embedding approach, we have greatly simplified the calculation by making use of manifest Lorentz invariance in our Ansatz for the WZ term.

Furthermore, we have explicitly shown that the reducibility between the constraints in the resulting gauge invariant Lagrangian comes from the absence of the Q^0 term, which naturally arises in analyzing the model’s symplectic structure. Requiring the gauge invariance of the total Lagrangian, we have successfully recovered it and thus have obtained the full gauge transformations consistently, while showing that all the zero modes solutions are not linearly independent, i.e., there exists a first-stage reducible trivial zero mode.

Similar to the symplectic Lagrangian embedding, we have shown in the BFT Hamiltonian embedding that the reducibility between the constraints is also related to the Q^0 component, and have successfully reconstructed the Lorentz covariant gauge embedded Lagrangian by using the delta functional of the Q^0 field in the path integral measure. However, in this scheme, which carries out the rigorous converting procedure from the second-class constraints to the effective first-class ones, due to the appearance of the constraints Ω_i , and the Ω_{ij} , in (4.6), originating from the symplectic structure of the theory, we cannot keep Lorentz covariance as seen in the new type of WZ Lagrangian. Nevertheless, we have shown that the total action including the NWZ action is invariant under the extended gauge transformations. Compared with this, since in the symplectic Lagrangian embedding we do not need to classify the constraints as second or first, we could freely require Lorentz covariance without any inconsistency. In short, a gauge embedded extension in the symplectic scheme is only related with the Lorentz covariant action while the usual symplectic scheme concerns only true constraints.

Finally, the method followed by this work is worthwhile in itself because there is no known systematic method yet except for the simplest one-form cases [12–14]. In this respect, we have in a new fashion generalized the symplectic Lagrangian embedding procedure to include highly non-trivial tensor fields which exhibit, for example, the existence of the first-stage reducible trivial zero mode among the others related to the reducibility of the theory.

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